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# Comparative Adequacy of Steady-State Versus Dynamic Models for Calculating Stockage Requirements

J. A. Muckstadt

November 1980

A Project AIR FORCE report
prepared for the
United States Air Force

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The formula for manually computing RRR WRSK items, which is
documented in AFLCR 57-18 (April 1979), should read:

$$RRR\ qty = [BRR \times BRC \times QPA] + \text{the greater of}$$
$$[DR \times QPA \times FH]\ \ \text{or}$$
$$[DDR \times QPA \times TFH]$$

where

$BRR$ = expected number of base level repairs (per
hundred flying hours),

$BRC$ = flying hours (in hundreds of hours) occurring
during a base repair cycle,

$QPA$ = number of units on an aircraft,

$DR$ = total demand rate at a base for an item
(demand per 100 flying hours),

$FH$ = flying hours (in hundreds of hours) in the
period of time in which the item's repair
facility is being set up,

$DDR$ = expected number of failed items per 100 hours
that cannot be repaired at base level and
will be evacuated to the depot for repair,

$TFH$ = total flying hours (in hundreds of hours) for
the support period.

This change does not affect the observations, findings, and conclusions
of the Report.

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See Reverse Side

Presents a two-echelon inventory model for
Air Force recoverable items in periods of
dynamic change in the demand process, such
as during initial provisioning or wartime.
Affirms that steady-state models should be
used only or mostly when flying activity is
relatively stable. The report also inves-
tigates the validity of a longstanding
assumption in the mathematics of inventory
systems: that depot delay in the resupply
of serviceable parts to a base is indepen-
dent of the number of units in base re-
supply. comparing the outputs of two
dynamic models, one embodying the assump-
tion and the other entailing meticulous
computations, the author concludes that the
assumption, although untrue, has a negligi-
ble effect on performance measurement and
stockage requirements, and that logisti-
cians therefore may freely embody it in
their models because of its mathematical
convenience.

R-2636-AF

# Comparative Adequacy of Steady-State Versus Dynamic Models for Calculating Stockage Requirements

J. A. Muckstadt

November 1980

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## PREFACE

For several years, Rand has investigated the ability of steady-state models to provide appropriate resource management in dynamic wartime environments, such as a NATO scenario. Of particular interest to Project AIR FORCE has been the modeling of inventory systems for aircraft parts when flying levels undergo large shifts. An effort has been made to devise means for estimating capability and computing stock-level requirements for such scenarios. Rand has also undertaken a rigorous evaluation of the effects on capability and requirements of certain widely used mathematical assumptions that have become wedded to classical inventory models, to see whether they are as valid as they are mathematically convenient.

This report investigates the importance of capturing the effects of changing levels of activity, presents a simple two-echelon dynamic model that achieves that purpose, and evaluates the possible misallocation of spares resulting from steady-state models. The report also examines the effects of a questionable independence assumption common to many models designed for capability assessment and requirements calculations. It demonstrates that assumptions regarding the independence of the process describing the number of units in base resupply and the depot repair process have an insignificant effect on the calculations. The study also demonstrates that the dynamic changes that would characterize a NATO wartime environment are important and are not adequately captured by steady-state models.

Several nonstationary models have been developed under Project AIR FORCE to deal with dynamic activity changes. The models described in this report represent one such effort. Readers interested in a dynamic model--currently under development and test--which is designed to examine a fairly wide range of combat scenarios are referred to Rand Note N-1482-AF, Model and Techniques for Recoverable Item Stockage When Demand and the Repair Processes are Nonstationary, by R. J. Hillestad and M. J. Carrillo. That Note describes a class of inventory models (Dyna-METRIC, formerly known as RAMS) that provide the user with the flexibility to compute support capability and requirements under wartime scenarios with changing levels of flying activity and fluctuating capabilities for repair.

The present study was performed for the Deputy Chief of Staff for Logistics and Engineering (AF/LE), Hq USAF, under two projects in the Project AIR FORCE Resource Management Program: "Concept Development and Project Formulation" and "Strategies to Improve Sortie Production in a Dynamic Wartime Environment." This study should be useful to Air Force analysts engaged in requirements determination and capability assessments at the Air Force Logistics Command, in AF/LE, and at the major air commands. More broadly, it should have utility for managers interested in applying dynamic allocation models to many kinds of inventory and stock-level problems.

## SUMMARY

This report presents a two-echelon inventory model for recoverable items when the demand process is nonstationary. The study is one product of Project AIR FORCE research conducted over the past several years that has questioned and investigated the ability of time-stationary (steady-state) models to provide appropriate resourcing in dynamic wartime environments. The report affirms the advisability of applying steady-state models only, or mostly, to periods of relatively stable flying activity--notably, peacetime flying programs. Dynamic models appear more promising in periods of dynamic change, such as initial provisioning and the early operational life of a weapon system and, more important, during wartime. In a NATO scenario, for example, wide swings in demand rates and repair rates are to be expected as flying levels fluctuate. In such a scenario, steady-state models are likely to cause significant misallocation of stock and miscalculation of the performance to be expected from the repair and supply systems.

The two-echelon model described here, like any model, is a mathematical simplification of the real world. In the course of demonstrating that steady-state models depart too far from realism, it was deemed advisable to investigate the validity of an assumption that has long been wedded to the mathematics of inventory systems and strongly affects requirements calculations: the assumption that depot delay in the resupply of serviceable parts to a base is independent of the number of units in base resupply (on order, in transit, or in repair). To evaluate the importance of this assumption, the outputs of two models

were compared. One model assumes a dynamic demand process, but assumes depot delay to be independent of the base resupply process. The other model assumes the same dynamic demand process but numerically solves the complex computational problem of evaluating the actual, dynamic, and conditional distributions. The conclusion was that this independence assumption has a negligible effect on performance measurement and stockage requirements, and that meticulous precision is therefore both unnecessary and computationally intractable--probably the major mathematical contribution of this report.

The inference, then, is that in many cases logisticians may freely proceed with models that embody this assumption because it is mathematically convenient, even though it is untrue.

This study presents some simplified illustrations of its two-echelon method for evaluating Air Force supply system performance when the demand process for recoverable items is nonstationary, such as occurs during wartime and initial provisioning. The examples illustrate some of the pitfalls of using a stationary representation of a nonstationary demand process in these situations to determine stockage requirements and to estimate expected system performance.

Two expressions are developed for the time-dependent probability distribution of the number of units in resupply at each location in a two-echelon resupply system. The resupply system is assumed to operate as follows. A nonstationary process generates item failures at the lower echelon locations, called "bases." The failed items are repaired either at the base or at the upper echelon, called a "depot." After issuing a unit to replace a failed item, the base inventory is replen-

ished by base maintenance whenever the failed item is repaired there, and by the depot otherwise. The organization repairing the item always exchanges a serviceable part for a broken one on a one-for-one basis. That is, the system follows a continuous review $(S-1, S)$ inventory policy, which the Air Force currently uses for recoverable items.

The study first develops an approximation to the probability distribution for the number of units in resupply at each location under the assumption that the demand process for a recoverable item is a nonstationary Poisson process at each base. This approximation is shown to be computationally tractable. Next, the study derives an exact representation for this distribution when the demand process is assumed to be a discrete time process related to the number of sorties flown during a particular time period. Because of computational difficulties, this distribution is shown to be of little value as an analysis tool on a large-scale basis; however, it provides a benchmark against which the approximating distribution can be tested. In all cases we examined, the approximation proved to be highly accurate, apparently equal to the one used in the METRIC model [1] for the same distribution when the demand process is assumed to be stationary.

## CONTENTS

## I. INTRODUCTION

Most probabilistic inventory models, including those that the Air Force currently uses, assume that the underlying demand process is stationary. Over time, however, the Air Force environment is not stationary. Flying activity for each type of aircraft increases rapidly when the aircraft is introduced into the active force and decreases as it is phased out of service. Correspondingly, demand rates for spare parts increase and then decrease. Nonetheless, stationary models are used to determine requirements for each item at each location throughout the aircraft's lifetime. Periodically, the values of daily demand rates, unit costs, shipping and repair times, procurement lead times, and the like, are adjusted to reflect current values, and new stock levels are calculated. These models are valuable during periods of relatively stable flying activity, such as those typical of peacetime. Whenever flying activity changes dramatically, however, the models can inaccurately estimate both stockage requirements and supply system performance. For example, when flying activity surges at the beginning of a war, these models provide little information concerning the logistics system's ability to support the increased flying activity. At certain points in time, stationary models either overstate or understate the capability to support a projected sortie rate.

This report describes a computationally tractable method that can be used to analyze the time-dependent behavior of a two-echelon inventory system for recoverable items (items amenable to repair when they fail). The system consists of a set of n locations, called bases, at

which flying occurs, and a centralized repair and inventory control point, called a depot. We assume the system operates in the following fashion. Each primary demand originates at one of the n bases. Upon failure of an item at a base, it is either repaired at that base or is sent to the depot for repair. If the failed item is repaired at the base, it is immediately entered into the base's maintenance system. Once the item has been repaired, it is sent to the base's supply organization and becomes available for issue. If the failed item is sent to the depot, the base immediately orders a replacement from the depot. The depot then immediately sends a replacement unit to the base, provided a serviceable spare unit is available; if it is not, the depot dispatches one to the base as soon as it becomes available. Thus, resupply of a base's supply organization comes from the base's maintenance organization when a failed item is repaired at a base and from the depot when the failed item is repaired there. (The possible flow of items in the system is displayed in Fig. 1.) In either case the organization performing the repair exchanges a serviceable item for a broken one as soon as pos-
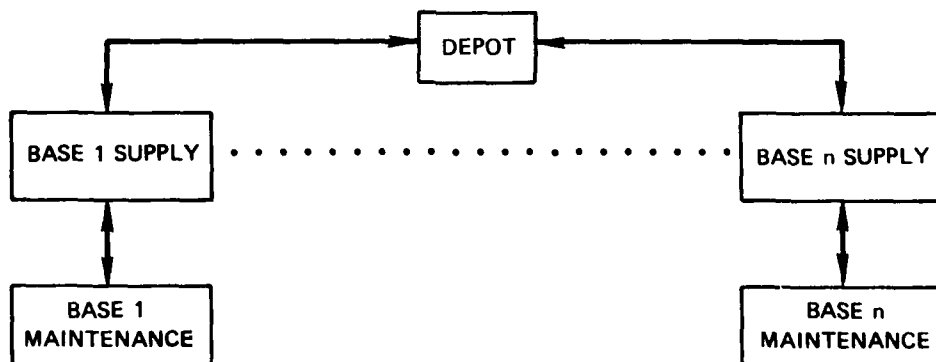


Fig. 1 — Possible item flow in the maintenance and supply system

sible. That is, the system follows an (S-1, S) continuous review policy, which is justified since most recoverable items are expensive and have low demand rates.

In subsequent sections we will derive, under different sets of assumptions, the time-dependent probability distribution of the number of units of a particular item in resupply at each location. The number of units in resupply at a base is the sum of those in base maintenance and those on order from the depot; the sum of those in depot resupply is the number of units in depot repair plus those en route to the depot from the bases. Once this distribution is known, time-dependent performance measures can be routinely calculated, such as ready rate, fill rate, Not Mission Capable--Supply (NMCS) rate, and the expected number of outstanding backorders at a point in time, t. It is also possible to find the minimum stock level[1] required at a location at time t to achieve a specified level of performance. Section II presents a simplified example of how this can be done.

Section II also demonstrates the effect of changing maintenance and transportation times, and illustrates the importance of using a nonstationary description of the demand process rather than a stationary approximation. Section II contains an example showing how the time-dependent demand process influences the number in resupply and the stock level needed to provide a given level of support.

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[1] At a base, the stock level measures on-hand serviceable inventory, plus items in repair at the base, plus items ordered from the depot that have not yet arrived, minus items backordered at the base. At the depot, the stock level measures the on-hand serviceable inventory, plus units in repair at the depot, plus units en route from the bases to the depot requiring depot repair, minus backordered items at the depot.

In Sec. III we assume that the demand process is a nonstationary Poisson process. Based on this and several other assumptions, we present an approximation to the time-dependent probability distribution, a nonstationary Poisson distribution, for the number of units in resupply at each location at time t. The approximation is similar to the one used by Sherbrooke [1] in his analysis of the same system when the demand process is a stationary Poisson process. In Sec. IV we develop, under a different set of assumptions, an alternative but exact expression for the same distribution. There we assume that the demand process is a discrete process. Specifically, we assume that a known number of sorties is flown each day, and the probability of an item failing on any sortie is p. The remaining assumptions made for the derivation of the distributions in these two sections are essentially the same.

As will be seen, the continuous time approximation developed in Sec. III is easy to evaluate computationally, whereas the distribution developed in Sec. IV is, for practical situations, intractable. Furthermore, as discussed in Sec. V, the quality of the approximating distribution is quite good when the chance or a backorder occurring at the depot is small (there is some depot safety stock) and/or the proportion of total failures requiring depot repair is small. In fact, the quality of the approximation is as good as the one used by Sherbrooke [1] to approximate the same distribution when the demand process is stationary.

Section VI briefly summarizes the report and discusses applications and policy implications.

## II. SOME ILLUSTRATIONS

We now illustrate the importance of taking a time-dependent view of the demand process. These examples consider the simplified case of a single base operating without depot support. The two-echelon calculations developed later are not used.

Suppose that an item has a fixed daily demand rate of .8 units, and a fixed base repair time of 5 days, and has all failures repaired at the base. A surge occurs in flying activity, after which the base repair time remains at 5 days and the base continues to perform all repairs; however, the demand rate following the increase in flying has the form $\lambda(t) = \alpha e^{-\beta t}$, where $\alpha = 3.16$ and $\beta = .1$. (Then the expected number of demands over the first 30 days following the time at which flying activity initially increased equals 30.)

The third column in Table 1 displays the value of

$$\mu_k = \int_{k-5}^{k} \lambda(t)dt,$$

the expected number of items in resupply at the end of day k. Suppose, for example, the stock level on each day is established such that the probability of having one or more backorders at the base is no greater than .2, assuming the demand process is a nonstationary Poisson process. (This policy is the one the Air Force uses to compute the stock level for spare aircraft engines, except that the demand process is assumed to be a stationary Poisson process [2,3].) For our example, the required

stock levels are given in the last column of Table 1. The stock level required on day k to meet this performance goal is the smallest nonnegative integer, s, such that

$$\sum_{n=0}^{s} \frac{e^{-\mu_k}(\mu_k)^n}{n!} \geq .8.$$

Observe that the peak requirement of 15 units occurs on day 5 and the minimum requirement for the 30-day period, 2 units, occurs on day 30. Furthermore, observe that the stock needed to achieve the specified level of service changes frequently.

Recall that the expected demand over the 30-day period is 30 units. If a stationary approximation to the demand process over the 30-day horizon is used and the total expected demand is 30 units, then the

Table 1

EXPECTED NUMBER OF UNITS IN RESUPPLY AND
REQUIRED STOCK LEVEL ON EACH DAY:
NONSTATIONARY DEMAND MODEL

| Day | Expected Demand | Expected Number of Units in Resupply | Stock Level Required to Achieve 0.8 Probability of No Backorders |
|-----|-----|-----|-----|
| 1 | 3.0 | 6.2 | 8 |
| 2 | 2.7 | 8.1 | 10 |
| 3 | 2.5 | 9.8 | 12 |
| 4 | 2.2 | 11.2 | 14 |
| 5 | 2.0 | 12.4 | 15 |
| 6 | 1.8 | 11.2 | 14 |
| 7 | 1.7 | 10.2 | 13 |
| 8 | 1.5 | 9.2 | 12 |
| 9 | 1.4 | 8.3 | 11 |
| 10 | 1.2 | 7.5 | 10 |
| 15 | .7 | 4.6 | 6 |
| 20 | .5 | 2.8 | 4 |
| 25 | .3 | 1.7 | 3 |
| 30 | .2 | 1.0 | 2 |

expected daily demand rate is 1 unit.[1] The expected number of units in resupply on each day under this assumption is given in column two of Table 2. Assuming the stock level is set so that the probability of having one or more backorders is no larger than .2 on any day, the corresponding minimum and maximum inventory requirements are estimated to be 6 units and 7 units, respectively. Thus the actual maximum requirement would be understated by 8 units. The level of support provided using the stationary demand model for determining stock levels changes substantially over the 30-day horizon. Table 3 gives the probability of having one or more backorders on each day. As indicated, supply support is inadequate during the early portion of the period and is much better than planned at the end of the period.

Reducing the resupply time is one way to reduce the requirement for spare stock or to increase the probability of satisfying all demands with a given stock level. To illustrate how a reduction in resupply

Table 2

EXPECTED NUMBER OF UNITS IN RESUPPLY AND
THE STOCK LEVEL ON EACH DAY:
STATIONARY DEMAND MODEL

| Day | Expected Demand | Expected Number of Units in Resupply | Stock Level Required to Achieve 0.8 Probability of No Backorders |
|-----|-----------------|--------------------------------------|------------------------------------------------------------------|
| 1 | 1.0 | 4.2 | 6 |
| 2 | 1.0 | 4.4 | 6 |
| 3 | 1.0 | 4.6 | 6 |
| 4 | 1.0 | 4.8 | 7 |
| 5 | 1.0 | 5.0 | 7 |

[1] The Air Force uses this method for computing demand rates during wartime--for example, to compute spare engine requirements [2,3].

Table 3

PROBABILITY OF HAVING ONE OR MORE
BACKORDERS WHEN USING THE STOCK
LEVELS COMPUTED USING THE STATIONARY
DEMAND MODEL IN THE DYNAMIC ENVIRONMENT

| Day | Probability | Day | Probability |
|-----|-------------|-----|-------------|
| 1 | .428 | 8 | .699 |
| 2 | .699 | 9 | .588 |
| 3 | .857 | 10 | .475 |
| 4 | .869 | 15 | .095 |
| 5 | .927 | 20 | .008 |
| 6 | .869 | 25 | .000 |
| 7 | .797 | 30 | .000 |

time affects performance, assume the base repair time is reduced to 3

days during peacetime and the first 10 days of the surge, and is 5 days

thereafter. Table 4 displays the expected number of units in resupply

Table 4

EXPECTED NUMBER OF UNITS IN RESUPPLY AND
THE REQUIRED STOCK LEVEL ON EACH DAY:
ALTERED BASE REPAIR TIME

| Day | Expected Number of Units in Resupply | Stock Level Required to Achieve 0.8 Probability of No Backorders |
|-----|------|------|
| 1 | 4.6 | 6 |
| 2 | 6.5 | 9 |
| 3 | 8.2 | 11 |
| 4 | 7.4 | 10 |
| 5 | 6.7 | 9 |
| 6 | 6.0 | 8 |
| 7 | 5.4 | 7 |
| 8 | 5.0 | 7 |
| 9 | 4.5 | 6 |
| 10 | 4.1 | 6 |
| 15 | 4.6 | 6 |
| 20 | 2.8 | 4 |

NOTE: Base repair time = 3 days through day 10,
and 5 days thereafter.

and the minimum stock level required to have at least a .8 probability of having no backorders on each day. Thus, by reducing the base repair time, the peak requirement for stock is reduced by slightly over 25 percent. We also note that to reduce the peak requirement, we must reduce the repair time for several days prior to the peak, not merely on the day the peak occurs.

The type of analysis used in this simplified illustration can be employed to investigate the impact of changing repair and transportation times on various system performance measures. The effect of time lags between increases (or decreases) in repair time or transportation times and the changes in performance (and the magnitude of the changes) can be examined using nonstationary models described in the following sections.

The examples discussed in this section are simplistic by design. They do not account for interactions between the depot and the bases. If one is content with this simplification, then it is easy to implement methods for finding stock levels for each location when the demand process is changing over time. The analysis becomes far more complex in the multiechelon case than in the single-location situation, since the straightforward technique employed in this section cannot be used to determine impact of having a given depot stock level on the expected number of units in base resupply at any point in time and, ultimately, on the time-dependent behavior of base supply performance. These complications and methods for dealing with them are discussed in the next two sections.

## III. A CONTINUOUS TIME MODEL

We begin this section by stating and discussing the major assumptions underlying the continuous and discrete time models developed here and in Sec. IV. We then derive the time-dependent probability distribution for the number of units in depot resupply and the approximate nonstationary probability distribution of the number of units in base resupply for the continuous demand process model.

### BASIC ASSUMPTIONS

For the model developed in this section, we assume that the demand process at each base in the system is a stationary Poisson process through time $t_0$, following which it becomes a nonstationary Poisson process. Thus, in this model the demand process is viewed as a continuous process. When $t \leq t_0$, the demand rate at base j is assumed to be a constant $\gamma_j$ units per day. Following $t_0$, we express the instantaneous demand rate at base j as $\lambda_j(t)$ which, we assume, does not depend on the number of units in repair. Thus we assume that the flying schedule is met regardless of inventory considerations. To simplify our notation, we assume, without loss of generality, that $t_0 = 0$.

We also assume that

1. Lateral resupply among bases is not permitted; however, stock levels can be changed over time at each location.

2. All failed parts are repaired.

3. Demand processes are independent from base to base.

4. All excess demand is backordered.

5. The echelon at which repair is performed depends only on the complexity of the repair. The probability of a failed unit at base j being repaired there is $r_j$.

6. There is no waiting or batching of items before starting repair on an item.

7. The repair time at base j is a constant $B_j$ days, and the depot repair cycle time, which includes the transportation time to the depot from a base, is a constant D days.

8. The transportation time from the depot to base j is a constant $A_j$ days. This transportation time includes the time it takes to place an order. Hence $A_j$ is the order and ship time for base j.

Before beginning the analysis, some clarifying comments concerning those assumptions will be helpful.

Assumption 1 is that lateral resupply is not allowed. That is, unplanned shipments between two bases to eliminate a temporary shortage are not allowed. The models developed here and in Sec. IV are designed to study the implications of certain supply, maintenance, transportation, and deployment policies in a dynamic environment. Stock levels are assumed to be specified in advance for each location at each point in time. Although these stock levels can be altered over time to account for planned changes in flying activity, real-time reallocation of assets among bases is not allowed. Consequently, no attempt is made to take advantage of the opportunities for improving system performance that might arise. The models are by design conservative; that is,

projected shortages could possibly be reduced by reallocating inventory
in real time when one location has a shortage while another has a con-
siderable amount of serviceable stock available. Other models can be
used to estimate the potential of lateral resupply as a means for
improving supply effectiveness (e.g., see Ref. 4).

We have also assumed that repair and transportation times are con-
stant. This assumption is unnecessary for the derivation given in this
section. Furthermore, these distributions can be time-dependent. These
extensions can be incorporated without significantly altering the
derivation we will give.

The derivation given in Sec. IV does require the repair and tran-
sportation times to be constant. A discrete time model can also be
derived using the same type of argument given in Sec. IV for the case
where the repair times and transportation times are constant but time-
dependent. However, the derivation of the distribution for the number
of units in resupply for the discrete time model is extremely complex
when repair and transportation times are assumed to be independent ran-
dom variables whose probability distributions have finite means.
Because the derivation is so long and complex, and the results are of
little practical value (because of the excessive computation required to
evaluate the probability distributions), we will not present the deriva-
tion.

The last assumption we will discuss in detail is the infinite
server assumption--Assumption 6. Clearly, in any practical situation
the number of available servers is always finite. However, empirical
evidence suggests that whenever the utilization rate is less than .7,

except for short periods of time, the infinite server approximation is reasonable [5]. This experimental evidence also indicates that the difference in the expected number of shortages at any point in time between the finite and infinite server models is insignificant when the utilization rate is less than or equal to .5. Consequently, the infinite server assumption appears to be reasonable unless the utilization rate is greater than .7 over an extended period of time.

Additional discussion of the assumptions is given in Ref. 1.


## DEPOT ANALYSIS

We begin the analysis by deriving the probability distribution for the number of units in resupply at the depot at any time $t > 0$. Let $N(t)$ represent the number of demands placed on the depot by all bases in $(0,t]$, and let $m(t)$ represent the expected number of demands placed on the depot during $(0,t]$; that is,

$$E[N(t)] = m(t) = \sum_{j=1}^{n} (1-r_j) \int_0^t \lambda_j(\tau) \, d\tau.$$

Next, let $M(t)$ represent the number of parts in depot repair at time t, let $M_1(t)$ represent the number of parts in depot repair at time t that were in repair at time 0, and let $M_2(t)$ denote the number of parts in depot repair at time t that enter the depot repair process following time 0. Thus $M(t) = M_1(t) + M_2(t)$, $t > 0$. We find the probability distribution for $M(t)$ by determining separately the distributions of $M_1(t)$ and $M_2(t)$ since both terms are independent.

Since the demand process is assumed to be a stationary Poisson process at each base through time 0, the depot demand process prior to time 0 is also a stationary Poisson process having rate

$$\gamma_0 = \sum_{j=1}^{n} (1-r_j) \, \gamma_j.$$

This is the case because, if a Poisson process with rate $\lambda$ is observed and each event is recorded with probability p, then the recorded process is a Poisson process with rate $p\lambda$ [4]. Consequently, $M_1(t)$ has a Poisson distribution with mean $\gamma_0 \cdot (D-t)$, $0 < t < D$, and $M_1(t) = 0$ with probability 1 when $t > D$.

Next, observe that $M_2(t)$ measures the number of units that fail at bases (that require depot repair) during $(0, t]$, when $0 < t < D$, and $(t-D, t]$, when $t > D$. But the failure processes at the bases are independent, nonstationary Poisson processes. Consequently, $M_2(t)$ has a nonstationary Poisson distribution, since the distribution of the sum of independent random variables each having a nonstationary Poisson distribution is again a nonstationary Poisson distribution. The mean of the distribution is $m(t)$, if $t \leq D$, and $m(t) - m(t-D)$, if $t > D$.

By combining the above results, we see that $M(t)$ is the sum of two independent Poisson processes (one stationary and the other nonstationary) when $0 < t \leq D$. Consequently, when $0 < t \leq D$, $M(t)$ is nonstationary Poisson distributed with expectation $\gamma_0 \cdot (D-t) + m(t)$. When $t > D$, $M(t) = M_2(t)$ and hence $M(t)$ again has a nonstationary Poisson distribution. In this case the mean is $m(t) - m(t-D)$.

(We note that a similar result can be obtained when D is not constant, but is a random variable whose distribution has a finite mean, D. In that case, M(t) also has a nonstationary Poisson distribution; however, its mean is not calculated in the same manner as when D is constant.)

As an example, suppose $\lambda_j(t) = \alpha_j e^{-\beta_j t}$, $t > 0$, a model often used to reflect flying activity during wartime. For this model,

$$m(t) = \sum_{j=1}^{n} (1-r_j) \; \alpha_j \int_0^t e^{-\beta_j x} \; dx = \sum_{j=1}^{n} (1-r_j) \; \alpha_j \; (1 - e^{-\beta_j t})/\beta_j,$$

and

$$E[M(t)] = \begin{cases} \displaystyle\sum_{j=1}^{n} (1-r_j) \; \alpha_j (1-e^{-\beta_j t})/\beta_j + \gamma_0(D-t), & t \le D \\[2em] \displaystyle\sum_{j=1}^{n} (1-r_j) \; \alpha_j \; e^{-\beta_j t} \, (e^{\beta_j D} - 1)/\beta_j, & t > D. \end{cases}$$

Ultimately, we are interested in determining the probability distribution for the number of units in resupply at each base at time t. As mentioned earlier, we will approximate this distribution with a nonstationary Poisson distribution. As we will see, this distribution's mean includes the expected delay experienced by a unit on order from the depot by a base at time t due to the unavailability of serviceable depot stock. Since an exact expression for this expected delay is unknown, we will approximate it using the well-known queueing formula $L = \lambda \cdot W$ as follows:

$$\begin{matrix} \text{Expected delay experienced} \\ \text{by an item on order by a} \\ \text{base at time t} \end{matrix} \approx \frac{\begin{matrix}\text{Expected depot} \\ \text{backorders at time t}\end{matrix}}{\begin{matrix}\text{Average depot arrival rate during} \\ \text{time t - D through time t}\end{matrix}}$$

Let $s_{0t}$ represent the depot stock level at time t, and $B_0(t; s_{0t})$ the expected number of backorders outstanding at the depot at time t. Then the (approximate) expected depot delay at time t is

$$B_0(t; s_{0t}) \bigg/ \left( \frac{1}{D} \int_{t-D}^{t} \lambda(\tau) \, d\tau \right) ,$$

where

$$\lambda(\tau) = \sum_{j=1}^{n} (1-r_j) \, \lambda_j(\tau)$$

is the depot arrival rate at time $\tau$. Furthermore,

$$B_0(t; s_{0t}) = \begin{cases} \sum_{x > s_{0t}} (x-s_{0t}) \, e^{-[\gamma_0(D-t)+m(t)]} [\gamma_0(D-t)+m(t)]^x/x!, & 0 < t \leq D, \\[4mm] \sum_{x > s_{0t}} (x-s_{0t}) \, e^{-[m(t)-m(t-D)]} [m(t) - m(t-D)]^x/x!, & t > D. \end{cases}$$

Observe that, under our assumptions, the expression for expected delay would be exact if the demand process were stationary.

## BASE ANALYSIS

We now derive an approximate probability distribution for the number of units in resupply for a base at time $t > 0$. Recall that the number of units in resupply at time t at a base is the sum of the items in base repair at time t and those items on order from the depot by the base at time t. The number of units in base repair at time t and the

number of units on order from the depot at time t are independent random
variables, since the random split of a Poisson process yields two
independent Poisson processes. Thus, we can find the distribution of
the number of units in resupply at a base by separately determining the
distribution of the number of units in base repair and the distribution
of the number of units on order from the depot by the base.

Let us first find the distribution of the number of units in base
repair at base j at time t. Let $X_j(t)$ represent this quantity. Using
the same argument as given for the depot, $X_j(t)$ is found to have a nons-
tationary Poisson distribution. If $0 < t < B_j$, the mean of this distri-
bution is

$$r_j \cdot \gamma_j \cdot (B_j - t) + m_j^1(t),$$

where, as before, $\gamma_j$ is the demand rate at base j at time $t \leq 0$, and

$$m_j^1(t) = r_j \int_0^t \lambda_j(\tau) \, d\tau.$$

If $t \geq B_j$, then the distribution's mean is $m_j^1(t) - m_j^1(t - B_j)$.

Next, we approximate the probability distribution for the number of
units on order from the depot by base j at time t with a nonstationary
Poisson distribution. (The reason for selecting the nonstationary Pois-
son distribution as the approximating distribution will be discussed
later on.) Consequently, all that needs to be done is to find an expres-
sion for the mean of the distribution at time t.

If $0 < t$ ⟨ , the mean of the distribution of the number of units
on order from the depot at time t by base j is the sum of (a) the

expected number of demands during $(0, t]$ at base $j$ requiring depot repair (i.e.,

$$m_j^2(t) = (1-r_j) \int_0^t \lambda_j(\tau) \, d\tau),$$

(b) the expected number of demands during $(t-A_j, 0]$ at base $j$ requiring depot repair (i.e., $\gamma_j \cdot (1-r_j) \cdot (A_j-t)$), and (c) the expected number of units backordered at the depot at time $t - A_j$ that correspond to orders placed by base $j$ (namely, $(1-r_j) \cdot \gamma_j \cdot$ expected backorders at the depot $/\gamma_0$

$$= (1-r_j) \cdot \gamma_j \cdot \sum_{x>s_0} (x-s_0) \, p(x|\gamma_0 D)/\gamma_0,$$

where $s_0$ is the depot stock level prior to time 0 and

$$p(x|\gamma_0 D) = e^{-\gamma_0 D} \frac{(\gamma_0 D)^x}{x!},$$

is the probability that $x$ units are in depot resupply). Thus, when $0 < t \le A$, the expected number of units on order from the depot by base $j$ that have not arrived by time $t$ is

$$m_j^2(t) + \gamma_j \cdot (1-r_j) \cdot (A_j-t + \sum_{x>s_0} (x-s_0) \, p(x|\gamma_0 D))/\gamma_0.$$

In a similar manner, we may find (approximately) the expected number of units on order from the depot by base $j$ when $t > A_j$. This expected value is the sum of (a) the expected number of demands during

the interval $(t-A_j, t]$ at base j requiring depot repair $(m_j^2(t) - m_j^2(t-A_j))$ and (b) the expected number of base j orders backordered at the depot at time t-A . This latter quantity can be determined as follows. Recall that

$$B_0(t-A_j; \ s_{0,t-A_j}) \Big/ \left( \frac{1}{D} \int_{t-D-A_j}^{t-A_j} \lambda(\tau)d\tau \right)$$

represents (approximately) the average delay experienced by a unit on order at the depot at time $t-A_j$. Multiplying this quantity by the average arrival rate from base j during $(t-D-A_j, t-D]$, we obtain an estimate of the number of units ordered from the depot by base j prior to $t-A_j$ but not received by time t. Then, combining these expressions, we have

$$m_j^2(t) = m_j^2(t-A_j) + \frac{\displaystyle\int_{t-D-A_j}^{t-A_j} (1-r_j) \ \lambda_j(\tau)d\tau}{\displaystyle\int_{t-D-A_j}^{t-A_j} \lambda(\tau)d\tau} \cdot B_0(t-A_j; s_{0,t-A_j})$$

as the mean of the nonstationary Poisson distribution of the number of units on order from the depot by base j at time $t > A_j$. If the value of $s_{0, \ t-A_j}$ is somewhat larger than the expected number in the depot repair cycle at time $t-A_j$, so that

$$B_0(t-A_j; \ s_{0,t-A_j}) \ \tilde{=} \ 0,$$

or if $r_j$ approaches 1, then the expected number of units being delayed at the depot for delivery to base j is approximately equal to 0. In either case, as we show in Sec. V, the nonstationary Poisson distribution is an excellent approximation to the distribution of the number of units on order from the depot by base j.

We have shown that the time-dependent distribution for the number of units in base repair is a nonstationary Poisson distribution, and that the time-dependent distribution for the number of units on order from the depot by a base can be approximated by a nonstationary Poisson distribution. These two distributions are independent, since the number of demands for depot resupply and base repair during any interval of time are independent random variables. (This is the case since the random split of a nonstationary Poisson process--the failure process at the base--results in these two independent nonstationary processes.) The number of units in resupply at the base is therefore the convolution of the random variables for the number of units in base repair and the number of units on order from the depot by the base. Consequently, the time-dependent distribution of the number of units in resupply at a base can be approximated by a nonstationary Poisson distribution.

## IV.  A DISCRETE-TIME, SORTIE-ORIENTED MODEL

We now develop, under a somewhat different set of assumptions, an exact, rather than an approximate, distribution for the number of units in resupply at time t at any location.  The approach taken here differs from the one used in Sec. III.  The major difference is that we now consider time to be divided into discrete increments, for example, into days.  On each day, the number of demands for spare stock for an item at a base depends on the known number of sorties flown that day.  We assume the probability is p that an item will fail during a sortie.  This assumption implies that the probability that an item fails on a sortie does not depend on either the base from which the sortie is flown or the type of sortie flown.  (This assumption is made only to simplify the analysis.  It can be dropped and the method we will use can be modified to develop the desired probability distributions; however, the computational burden increases substantially.) We also assume that each aircraft contains only one unit of the item.  Our analysis can easily be extended to systems containing more than one unit per aircraft.  We continue to make Assumptions 1 to 8 listed in Sec. III.  In addition, we assume that the probability of base repair is the same at all bases, i.e., $r_j = r$, although this assumption is not crucial and is made only to simplify the presentation.  Lastly, we assume that $A_j \geq B_j$; that is, the depot-to-base transportation time is at least as large as the base repair time.  This assumption is satisfied in most instances in the Air Force.  For example, the base repair time for F-15 avionics items is normally only several days, whereas the depot-to-base order time plus

transportation time is usually 10 or more days. Again, this assumption could be removed without changing the method used to compute the desired distributions.

The number of units in resupply on day k at base j is the number of units ordered from the depot by base j by day $k-A_j$ that have not arrived at base j by day k, denoted by $W_{j1}(k)$, plus all units ordered by base j from the depot during days $k-A_j + 1$ through $k-B_j$, denoted by $W_{j2}(k)$, plus all demands occurring at base j on days $k-B_j + 1$ through k, denoted by $W_{j3}(k)$. Since the random variables $W_{j1}(k)$, $W_{j2}(k)$, and $W_{j3}(k)$ measure the number of units in resupply that are due to demands occurring in non-overlapping intervals of time (see Fig. 2), and since the demand on any day is independent of that on any other day (we assume that aircraft are available to fly the scheduled sorties), these random variables are independent. Thus we can find the distribution for the number of units in resupply at base j on day k by determining separately the distributions of $W_{j1}(k)$, $W_{j2}(k)$, and $W_{j3}(k)$ and taking the convolution of these three distributions.
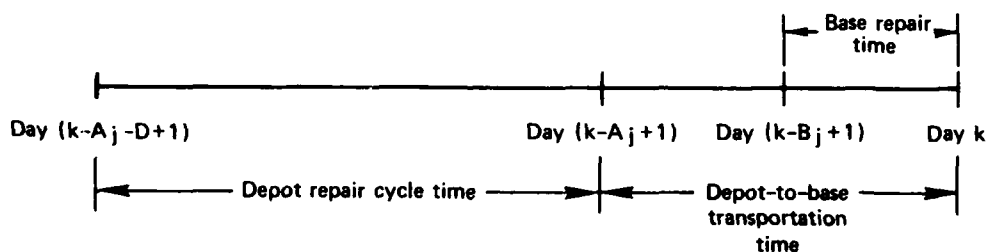


Fig. 2 — Time sequence at base j

Before we derive these distributions, let us introduce some new nomenclature and make some observations. Let

$X_{kj}$ = the number of sorties flown on day k at base j,

$X_{k0}$ = the number of sorties flown on day k at all other bases,

$X_k$ = $X_{kj} + X_{k0}$ = the total sorties flown on day k,

$Y_{k0}$ = the number of failures at base j on day k requiring depot repair,

$Y_{kj}$ = the number of failures at all other bases on day k requiring depot repair,

$Y_k$ = $Y_{kj} + Y_{k0}$, and

q = (1-r)p.

Since there are $X_k$ sorties flown on day k, each of which generates a failure requiring depot repair with probability q, $Y_k$ has a binomial distribution, that is,

$$P(Y_k = y) = \begin{cases} \binom{X_k}{y} \cdot q^u \cdot (1-q)^{X_k-y} & , y=0, \ldots, X_k, \\ \\ 0 & , \text{otherwise} \end{cases}$$

We assume that demands resulting from sorties flown on day k are entered into the depot or base repair cycle at the end of day k. All repairs are assumed to be completed at the end of a day.

Consequently, the probability that y units are in the depot repair cycle just after the end of day k is

$$P\left(\sum_i Y_i = y\right) = \begin{cases} \binom{\sum_i X_i}{y} q^y (1-q)^{\left(\sum_i X_i - y\right)}, & y=0, 1, \ldots, \sum_i X_i, \\ \\ 0 & , \text{otherwise.} \end{cases}$$

where the index of summation i ranges from k-D+1 through k.

Let us first derive $P(W_{j1}(k) = w)$. Observe that any unit on order from the depot by base j prior to day $k-A_j-D + 1$ must have been satisfied by the end of day k, assuming that a first-come first-serve policy is followed. Consequently, $W_{j1}(k)$ measures the number of depot orders placed by base j on days $k-A_j-D + 1$ through $k-A_j$ that are not satisfied by day k. Let $s_0$ represent the depot stock level on day $k-A_j$. We will find $P(W_{j1}(k) = w)$ by determining

$$P(W_{j1}(k) = w \mid \sum_i Y_i = s_0 + y),$$

where the index of summation, i, ranges from $i = k-A_j-D+1$ through $i = k-A_j$.

Next, observe that if some orders placed on the depot by base j on days $k-A_j-D+1$ through $k-A_j$ are not satisfied by day k, then the number of failures on days $k-A_j-D+1$ through $k-A_j$ at all bases that require depot repair must exceed the depot stock available to meet those demands. That is, if $w \geq 1$, then

$$\sum_{i=k-D-A_j+1}^{k-A_j} Y_i > s_0$$

Consequently, there exists a first day among the days $k-D-A_j+1$ through $k-A_j$ when the total depot demand over this period exceeds $s_0$. Let L be the random variable denoting this day. By conditioning on L we see that

$$P(W_{j1}(k) = w \mid \sum_i Y_i = s_0 + y) = \sum_{\ell=k-D-A_j+1}^{k-A_j} P(W_{j1}(k)$$

$$= w \mid L=\ell; \sum_i Y_i = s_0 + y) \cdot P(L=\ell \mid \sum_i = s_0 + y).$$

However, $L = \ell$ if and only if the total demand on days $k-D-A_j+1$ through $\ell-1$ does not exceed $s_0$ and the demand on day $\ell$ is sufficient to raise the total demand for days $k-D-A_j + 1$ through $\ell$ above $s_0$. Thus

$$P(W_{j1}(k) = w \mid \sum_i Y_i = s_0 + y)$$

$$= \sum_{\ell=k-A_j-D+1}^{k-A_j} \sum_{a=\max(0,s_0-X_\ell)}^{s_0} \sum_{e=s_0-a+1}^{X_\ell}$$

$$P\left(W_{j1}(k) = w \mid \sum_{i=k-A_j-D+1}^{\ell-1} Y_i=a; \ Y_\ell=e; \ \sum_{i=k-A_j-D+1}^{k-A_j} Y_i=s_0+y\right)$$

$$P\left(\sum_{i=k-A_j-D+1}^{\ell-1} Y_i=a; \ Y_\ell=e \ \Big| \ \sum_{i=k-A_j-D+1}^{k-A_j} Y_i=s_0+y\right).$$

Also, since the probability of a failure on a sortie is independent of the number of failures occurring on all other sorties, we see that

$$P\left(\sum_{i=k-A_j-D+1}^{\ell-1} Y_i = a;\ Y_\ell = e \ \middle|\ \sum_{i=k-A_j-D+1}^{k-A_j} Y_i = s_0 + y\right)$$

$$= \frac{P\left(\displaystyle\sum_{i=k-A_j-D+1}^{\ell-1} Y_i = a;\ Y_\ell = e;\ \sum_{i=k-A_j-D+1}^{k-A_j} Y_i = s_0 + y - (a+e)\right)}{P\left(\displaystyle\sum_{i=k-A_j-D+1}^{k-A_j} Y_i = s_0 + y\right)}$$

$$= \frac{\displaystyle\binom{\sum_{i=k-A_j-D+1}^{\ell-1} X_i}{a}\binom{X_\ell}{e}\binom{\sum_{i=\ell+1}^{k-A_j} X_i}{s_0+y-(a+e)}}{\displaystyle\binom{\sum_{i=k-A_j-D+1}^{k-A_j} X_i}{s_0+y}}$$

whenever $k-A_j-D+1 < \ell < k-A_j$. The obvious adjustments must be made for the cases where $\ell = k-A_j-D+1$ and $\ell = k-A_j$.

Observe that if $W_{j1}(k) = w$, then some of these units could be demanded from the depot by base j on day $\ell$, where, as before, $\ell$ is the first day among the days $k-A_j-D+1$ through $k-A_j$ when the total depot demand exceeds $s_0$, and the remainder of these units are demanded at base j during days $\ell+1$ through $k-A_j$. Let $Z_{j\ell}$ represent the number of units ordered from the depot by base j on day $\ell$ that are unsatisfied by day k. Note that

$$\sum_{i=\ell+1}^{k-A_j} Y_{ij}$$

measures the number of units that fail at base j on days $\ell+1$ through $k-A$ that require depot repair. Then

$$P\left(W_{j1}=w \;\middle|\; \sum_{i=k-A_j-D+1}^{\ell-1} Y_i = a; \; Y_\ell=e; \; \sum_{i=k-A_j-D+1}^{k-A_j} Y_i = s_0+y\right)$$

$$= \sum_{f=0}^{\min(e,w)} P\left(Z_{j\ell}=f; \; \sum_{i=\ell+1}^{k-A_j} Y_{ij}\right.$$

$$= w-f \;\middle|\; \sum_{i=k-A_j-D+1}^{\ell-1} Y_i=a; Y_\ell=e; \; \sum_{i=k-A_j-D+1}^{k-A_j} Y_i=s_0+y\right)$$

$$= \sum_{f=0}^{\min(e,w)} P\left(Z_{j\ell}=f \;\middle|\; \sum_{i=k-A_j-D+1}^{\ell-1} Y_i=a; Y_\ell=e; \; \sum_{i=k-A_j-D+1}^{k-A_j} Y_i=s_0+y\right)$$

$$P\left(\sum_{i=\ell+1}^{k-A_j} Y_{ij}=w-f \;\middle|\; \sum_{i=k-A_j-D+1}^{\ell-1} Y_i=a; Y_\ell=e; \; \sum_{i=k-A_j-D+1}^{k-A_j} Y_i=s_0+y\right)$$

Note that $Z_{j\ell}$ depends only on the number of failures at all bases requiring depot repair on days $k-A_j-D+1$ through $\ell$, and that

$$\sum_{i=\ell+1}^{k-A_j} Y_{ij}$$

depends only on the number of failures requiring depot repair on days $\ell+1$ through day $k-A_j$. The latter implies that

$$P\left(\sum_{i=\ell+1}^{k-A_j} Y_{ij}=w-f \;\middle|\; \sum_{i=k-A_j-D+1}^{\ell-1} Y_i=a; Y_\ell=e; \; \sum_{i=k-A_j-D+1}^{k-A_j} Y_i=s_0+y\right)$$

$$= P\left( \sum_{i=\ell+1}^{k-A_j} Y_{ij} = w-f \;\middle|\; \sum_{i=\ell+1}^{k-A_j} Y_i = s_0 + y - (a+e) \right)$$

$$= \frac{\left( \displaystyle\sum_{i=\ell+1}^{k-A_j} X_{ij} \right)_{w-f} \left( \displaystyle\sum_{i=\ell+1}^{k-A_j} X_{i0} \right)_{s_0+y-(a+e)-(w-f)}}{\left( \displaystyle\sum_{i=\ell+1}^{k-A_j} X_i \right)_{s_0+y-(a+e)}} \quad,$$

and the former implies that

$$P\left( Z_{j\ell} = f \;\middle|\; \sum_{i=k-A_j-D+1}^{\ell-1} Y_i = a; \; Y_\ell = e; \; \sum_{i=k-A_j-D+1}^{k-A_j} Y_i = s_0 + y \right)$$

$$= \sum_{g=f}^{\min(X_{\ell j},e,w)} P\left( Z_{j\ell} = f \;\middle|\; \sum_{i=k-A_j-D+1}^{\ell-1} Y_i = a; \; Y_\ell = e; \; Y_{\ell j} = g \right) \cdot P(Y_{\ell j} = g \,|\, Y_\ell = e)$$

$$= \sum_{g=f}^{\min(X_{\ell j},e,w)} \frac{\binom{g}{f} \cdot \binom{e-g}{a+e-s_0-f}}{\binom{e}{a+e-s_0}} \cdot \frac{\binom{X_{\ell j}}{g} \cdot \binom{X_{\ell 0}}{e-g}}{\binom{X_\ell}{e}}$$

$$= \sum_{g=f}^{\min(X_{\ell j},e,w)} \frac{\binom{a+e-s_0}{f} \cdot \binom{s_0-a}{g-f} \cdot \binom{X_\ell-e}{X_{\ell j}-g}}{\binom{X_\ell}{X_{\ell j}}}$$

By combining the above results we can determine

$$P(W_{j1}(k) = w \mid \sum_{i=k-A_j-D+1}^{k-A_j} Y_i = s_0 + y).$$

Since, under our assumptions,

$$\sum_{i=k-A_j-D+1}^{k-A_j} Y_i$$

has a binomial distribution as noted previously, we can evaluate $P(W_{j1}(k) = w)$ using

$$P(W_{j1}(k) = w) = \sum_{y \geq w} P\left(W_{j1}(k) = w \mid \sum_{i=k-A_j-D+1}^{k-A_j} Y_i = s_0 + y\right) \cdot P\left(\sum_{i=k-A_j-D+1}^{k-A_j} Y_i = s_0 + y\right), \quad w \geq 1,$$

and

$$P(W_{j1}(k) = 0) = 1 - \sum_{w=1}^{\infty} P(W_{j1}(k) = w).$$

Next, let us establish the probability distributions for the random variables $W_{j2}(k)$ (the number of failures at base $j$ during days $k-A_j+1$ through $k-B_j$ that require depot repair) and $W_{j3}(k)$ (the number of failures at base $j$ on days $k-B_j+1$ through $k$). Since the probability that a failure will occur on a sortie does not depend on the number of failures occurring on other sorties, $W_{j2}(k)$ and $W_{j3}(k)$ are independent and binomially distributed. In particular, $W_{j2}(k)$ has a binomial distribution with parameters

$$\sum_{i=k-A_j+1}^{k-B_j} X_{ij}$$

and q, and $W_{j3}(k)$ has a binomial distribution with parameters

$$\sum_{i=k-B_j+1}^{k} X_{ij}$$

and p.

Recall that the number of units in resupply on day k at base j, $W_j$ (k), is the sum of $W_{j1}(k)$, $W_{j2}(k)$, and $W_{j3}(k)$. We also have established that these three random variables are independent. Since we have shown how to determine the distribution for $W_{j1}(k)$, $W_{j2}(k)$, and $W_{j3}(k)$, the distribution for $W_j(k)$ is the convolution of these three distributions.

## V.  A COMPARISON OF THE EXACT AND APPROXIMATE MODELS

In previous sections we developed two representations of the probability distribution of the number of units in resupply at each location based on different assumptions concerning the nature of the demand process.  Furthermore, the probability distribution obtained in Sec. III is only an approximation to the actual distribution of the number of units in resupply at a base.

To test the accuracy of the approximation, we compared distributions obtained using this approximation with those calculated using the exact expression developed in Sec. IV.  A sample of eight avionics items found on the F-15 weapon system was selected to make the comparison. These items were chosen to represent various combinations of an item's failure rate and its value of r, the probability that it is repaired at a base.  Items were chosen that have low, medium, and high failure rates, and values of r ranging from .05 to .95.  (The experiment was essentially a 3- experiment in which all combinations of failure rates (low, medium, high) and r values (low, medium, high) were represented.)

The demand models were selected so that a substantial degree of nonstationarity was present.  The models reflected a sharp increase in flying activity at each base following a long period during which flying activity was constant.  After the initial surge, flying activity was assumed to decrease exponentially at each base so that 30 days later it had returned to approximately the stationary value that preceded the surge.  (See Fig. 3 for an illustration.)
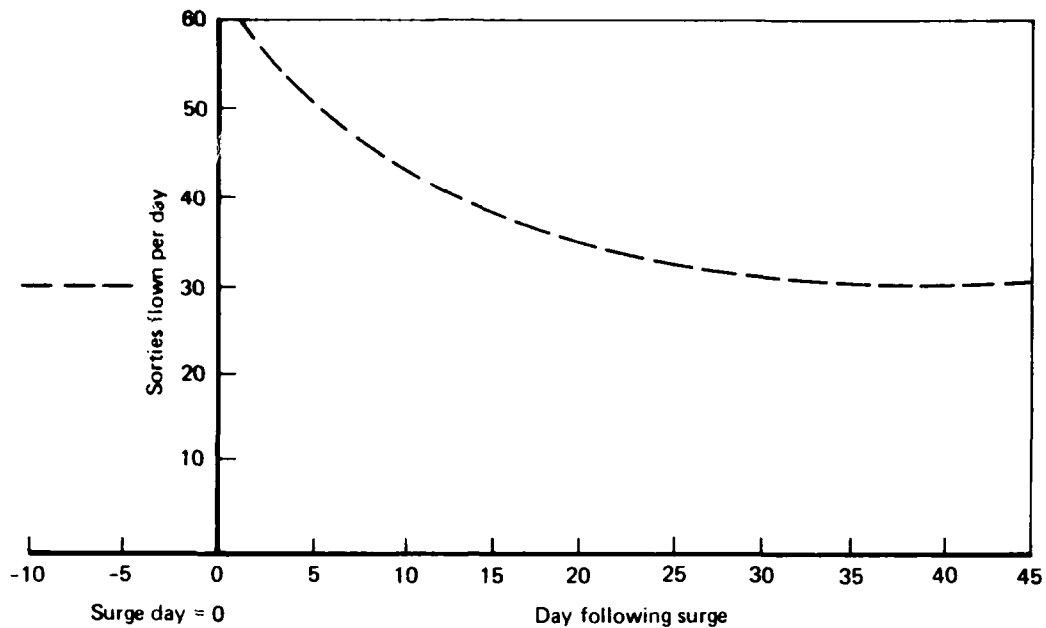
Fig. 3--Daily flying activity at the base for which the
exact and approximate distributions were calculated

The base repair times, $B_j$, order and ship times, $A_j$, and depot
repair cycle times, D, were 4 days, 10 days, and 40 days for each item,
respectively.

Using these data, we compared the approximate and exact distribu-
tions for days 5, 15, and 30 following the surge in flying. First, we
assumed the depot had no safety stock, that is, we assumed the depot
stock was approximately equal to the expected number of units in the
depot repair cycle. To be precise, we set the depot stock level for
each day so that it was equal to the smallest integer that was greater
than or equal to the expected number of units in the depot repair cycle
at that time. This meets current Air Force policy, which requires that
the depot stock level should be at least as large as the expected number
of units in the depot repair cycle [6]. This comparison is a "worst

case" type of comparison; if the depot stock is larger than this lower bound, the quality of the approximation must improve. This fact is subsequently illustrated. Furthermore, note that as the depot stock level approaches infinity,

$$P(W_{j1}(k) = 0) \rightarrow 1.$$

Consequently, the number of units in resupply at base j on day k when $S_0$ = $\infty$ is $W_{j2}(k) = W_{j3}(k)$. Recall that $W_{j2}(k)$ and $W_{j3}(k)$ are independent random variables and each has a binomial distribution. If the length of the time period used in the exact model approaches zero, then each of these binomial distributions approaches a Poisson distribution. Thus if $S_0 \rightarrow \infty$ and the length of the time period in the exact model approaches zero, the exact distribution approaches a Poisson distribution (since the sum of two independent Poisson random variables is a Poisson random variable) and the exact and approximate distributions become identical.

In the experiment we also assumed that the flying activity at the base for which the exact and approximate distributions were explicitly calculated was one-half the system's total. This again is a worst-case type of comparison. By having a large fraction of activity concentrated at a single base, we increase the effect that the depot stock level has on the distribution of the number of units in resupply at that base. If the flying activity at a base is small in comparison with the total system flying activity, then depot backorders have a minimal effect on the expected number of units in resupply at the base. The effect of depot delay in satisfying base orders on the quality of the approximation increases as the fraction of depot demands attributable to the base

increases. Thus, the experiment was designed to test the quality of the approximating distribution under reasonably extreme conditions.

The results of the experiment showed that in all cases the nonstationary Poisson distribution is an excellent approximation to the exact distribution of the number of units in resupply at a base, even when the depot stock level is at its lower bound. The distributions generally agree to two decimal places. Table 5 compares the approximate and exact distributions for three items on days 5, 15, and 30, for various values of r and p. The quality of the approximation also appeared to be affected by the items' failure rate for similar values of r; the quality of the approximation improved slightly as the failure rate decreased.

To illustrate the effect of the depot stock level on the quality of the approximation, we next increased its value. The effect of this increase is illustrated in Table 6. As the example demonstrates, the quality of the approximation always improves as the depot stock level increases.

This experiment has shown that a nonstationary Poisson distribution provides an excellent approximation to the distribution of the number of units in resupply at a base under certain conditions. We believe that in several respects the test performed was a "worst case" type of test. There was a substantial initial increase in flying activity per day followed by a rapid decline in flying activity, the depot stock level was low, and the base examined produced half the depot demand. Although we have obviously performed a limited test, the results indicate that in realistic situations a nonstationary Poisson distribution can safely be used to represent the distribution of the number of units in resupply.

Table 5

COMPARISON OF THE EXACT AND APPROXIMATE
DISTRIBUTIONS OF THE NUMBER OF UNITS IN
BASE RESUPPLY WHEN THE DEPOT STOCK LEVEL
EQUALS THE SMALLEST INTEGER GREATER THAN
OR EQUAL TO THE EXPECTED NUMBER OF
UNITS IN DEPOT RESUPPLY

| Day | No. of Units in Base Resupply | Probability Distribution | |
|-----|---------|-------------|-------|
| | | Approximate | Exact |
| r = .05, p = .0001 | | | |
| Day 5 | 0 | .959 | .954 |
| | 1 | .040 | .045 |
| | 2 | .001 | .001 |
| Day 15 | 0 | .960 | .953 |
| | 1 | .039 | .046 |
| | 2 | .001 | .001 |
| Day 30 | 0 | .969 | .960 |
| | 1 | .031 | .039 |
| | 2 | .001 | .001 |
| r = .96, p = .01 | | | |
| Day 5 | 0 | .108 | .101 |
| | 1 | .240 | .231 |
| | 2 | .267 | .266 |
| | 3 | .199 | .203 |
| | 4 | .111 | .203 |
| | 5 | .049 | .054 |
| | 6 | .018 | .020 |
| Day 15 | 0 | .182 | .020 |
| | 1 | .310 | .314 |
| | 2 | .264 | .263 |
| | 3 | .150 | .147 |
| | 4 | .064 | .061 |
| | 5 | .022 | .020 |
| Day 30 | 0 | .244 | .248 |
| | 1 | .344 | .346 |
| | 2 | .243 | .241 |
| | 3 | .114 | .112 |
| | 4 | .040 | .039 |
| | 5 | .011 | .011 |
| r = .5, p = .0001 | | | |
| Day 5 | 0 | .969 | .967 |
| | 1 | .030 | .032 |
| | 2 | .001 | .001 |
| Day 15 | 0 | .972 | .970 |
| | 1 | .027 | .029 |
| | 2 | .001 | .001 |
| Day 30 | 0 | .978 | .975 |
| | 1 | .022 | .025 |

Table 6

COMPARISON OF THE EXACT AND APPROXIMATE DISTRIBUTIONS OF
THE NUMBER OF UNITS IN BASE RESUPPLY ON DAY 15
WHEN r = .6 AND p = .002

| Depot Stock Level Equals the Smallest Integer that is Greater Than or Equal to: | No. of Units in Base Resupply | Probability Distribution | |
|---|---|---|---|
| | | Approximate | Exact |
| Expected number of units in depot resupply ($s_0$ = 3) | 0 | .532 | .573 |
| | 1 | .336 | .315 |
| | 2 | .106 | .090 |
| | 3 | .022 | .018 |
| | 4 | .004 | .003 |
| Expected number of units in depot resupply plus 1 ($s_0$ = 4) | 0 | .571 | .586 |
| | 1 | .320 | .312 |
| | 2 | .090 | .084 |
| | 3 | .017 | .016 |

We also observed that the computation time needed to compute the exact distribution ranged from roughly 500 to 2000 times the amount needed to calculate the approximate distribution. Considerably less than a second of CPU time (on an IBM 370/168) was required to compute the approximate distribution in all cases. Furthermore, many computational problems (both roundoff and underflow difficulties) frequently arose during the calculation of the exact distribution. Depending on the values assumed by the distribution's parameters, different approaches had to be taken to compute the exact distribution. Because of the large amount of time needed to perform these calculations and the numerical problems that are present, the exact representation of the probability distribution for the number of units in resupply cannot be used in studies involving large numbers of items. It will not be of practical use in an Air Force requirements computation system for the same reasons.

In Ref. 1, Sherbrooke analyzed the same system that we have examined under the assumption that the demand process is a stationary Poisson or compound Poisson process. His development closely parallels the

one given in Sec. III. The expression he develops for the distribution
of the number of units in resupply is an approximation. Shanker [7]
compares Sherbrooke's approximation with the exact one that he develops.
He shows that when the depot stock level exceeds the expected number in
the depot repair cycle, Sherbrooke's approximation is quite good. We
note that Sherbrooke's approximation is widely used in practice (for
example, it is the basis for the Air Force's Variable Safety Level
method [6,7]). Based on the comparison we have made and the ones
reported by Shanker, we believe that the approximation developed in Sec.
III is as accurate as the one developed by Sherbrooke.

## VI. CONCLUDING REMARKS

Our primary objective was to demonstrate that methods are available that can be easily used to help assess the Air Force's ability to support its flying mission during a surge in flying activity. Two approaches were taken to derive, under different sets of assumptions, the probability distribution of the number of units in resupply at each location in the two-echelon system at any time t. As was shown, one is an approximation to this distribution while the other is an exact representation. Furthermore, we have shown that the approximation is easily computed and closely approximates the exact distribution; the exact distribution is computationally intractable and of little practical value.

To this point our attention has focused on developing these methods for evaluating supply system performance. There are many potential applications of these models to the analysis of the effect of transportation and maintenance policies on supply system performance. However, we will discuss the principal applications of the nonstationary demand models in the areas of stock-level determination for spare recoverable items in a dynamic, short-time-horizon, wartime environment and during initial provisioning.

Planners are constantly faced with the problem of determining what amount of inventory should be prepositioned as war readiness stock and what airlift capabilities should be provided so as to achieve a specified level of supply effectiveness for a short-time-horizon armed conflict. One of the main applications of this nonstationary probability

distribution is to help answer these questions. Since the models measure the probability of having a specific number of shortages for each item at any time t for given item stock levels, the stock levels required to achieve any desired level of supply effectiveness can be calculated for each item at any point in time. For example, suppose a required level of supply support is established for a group of items for each day in the planning horizon. Furthermore, assume that the first time when additional inventory can be provided from another location--say, via airlift--is on day $n_1$, the second time on day $n_2$, and the last time on day $n_m$. Then the time horizon can be subdivided into $n_m + 1$ periods whose lengths correspond to the times between successive arrivals of additional inventory. If we also know the desired supply support goals for each period, the times at which additional supplies can arrive, the maximum amount of stock that can be received (or shipped) in each delivery, and the expected number of parts to be consumed on each day, then we can use nonstationary demand models to establish the quantity of each item that should be prepositioned and shipped in each delivery.

Several models can be formulated for calculating the desired dynamic stock levels. The models can be quite simple or rather complex, depending on the choice of the objective function and constraints. For example, if a supply-effectiveness constraint is stated by item, such as that the fill rate must be at least .8 on each day, and the goal is to provide the minimum amount of inventory to have on hand each day to meet this constraint, then the stock levels can be found in a relatively straightforward manner. If the objective is to minimize inventory

investment over all items while satisfying constraints on the expected

number of serviceable aircraft available on each day, the problem

becomes much more difficult. Nonetheless, for any choice of objective

function and constraints, the optimal stock levels could be computed for

each item in each period, once the dynamic models have been used to cal-

culate nonstationary distribution of the number of units in resupply at

each location.

Dynamic models can also be used for explicitly examining the impli-

cations of transportation policy--the number and timing of deliveries of

additional inventory--on supply-effectiveness and cost. The amount of

prepositioned stock that is needed can be calculated as a function of

the transportation policy selected. Thus the proper balance between

inventory costs and transportation costs can be established to achieve

any desired level of performance. As a consequence, inventory policy--

for example, for WRSK and BLSS--and transportation policy for supplying

inventory to a theater of war can be established recognizing the

existent interactions.

The current Air Force policy for determining requirements for war

reserve material is stated in AFLCR 57-18 [8]. The methods described in

this regulation for computing stock levels take into account whether an

item has a Remove-and-Replace (RR) or a Remove-Replace-Repair (RRR)

maintenance concept, whether an item is a Line Replaceable Unit (LRU) or

a Shop Replaceable Unit (SRU), and whether the unit is primarily

repaired at a base or a depot. For example, the formula for computing

the stock level for an LRU that is repaired primarily at a base follow-

ing an RRR maintenance concept is:

$$\text{Quantity} = DR \times QPA \times FH + BRR \times BRC \times QPA,$$

where

DR = total demand rate at a base for an item
(demands per 100 flying hours),

QPA = number of units of the item on an aircraft,

FH = flying hours (in hundreds of hours) in the
period of time in which the item's repair
facility is being set up,

BRR = expected number of base level repairs (in
hundreds of flying hours), and

BRC = flying hours (in hundreds of hours) occurring
during a base repair cycle.

Observe that the stock level is computed using an expected-value type of calculation. No attempt is made to take the uncertainty of demand into account. Potential supply performance on each day cannot easily be determined using the current Air Force methodology. Note that the choice of the BRC factor affects the stock level significantly. There is no guarantee that the proper action is to set BRC so that the term BRR x BRC x QPA is as large as possible. Because this peak requirement may occur for only a small fraction of the planning horizon, a large investment in the item may be undesirable. In fact, it is impossible to ascertain what value BRC should assume without performing the type of analysis discussed in Secs. III and IV. Furthermore, the interaction between the depot and bases is ignored entirely in these calculations. In short, the current policy has some serious defects.

The inadequacy of the present policy has been recognized by Air Force planners. As a result, a revision to AFLCR 57-18 is being

prepared which these planners believe will address its major deficiencies. The following is a brief description of the revised method [9].

Stock levels are first calculated for each item using the present Air Force technique. These stock levels are made as large as possible by finding the period during which the expected number of units in base resupply is at its maximum. For example, for an LRU repaired primarily at a base using an RRR maintenance concept, the stock level is maximized by selecting the period during which the BRC factor is the largest. This resulting quantity can be further adjusted by multiplying it by a factor to yield some safety stock.

Using these stock levels, the expected number of backorders and the expected number of serviceable aircraft are computed. These performance measures are calculated under the assumption that the number of units in base resupply for each item is Poisson distributed. The mean of this distribution is assumed to be equal to the maximum expected number of units in resupply during the planning horizon. Once the expected performance levels have been determined, a heuristic, which is a gradient type of procedure, is used to find new values for the stock levels. These stock levels are selected in the hope of achieving these two performance levels at minimum cost.

There are drawbacks to the revision of the current Air Force method for computing war readiness stocks that we have discussed. First, it does not consider the dynamic nature of the problem. Setting the mean of the Poisson distribution describing the number of units in base resupply to its largest value over the planning horizon makes the dynamic problem appear to be static. It is unknown whether the maximum mean

occurs for a significant fraction or a very small fraction of the planning horizon. The differences between the maximum and minimum values of the mean are not considered. The actual mean value of the Poisson distribution over the planning horizon should be used in setting stock levels. Otherwise, serious errors can occur when setting item stock levels. Furthermore, the system's dynamic performance cannot be readily examined.

A second drawback is that the revised method does not accurately take into account the effect of depot stock on base level performance. Consequently, it cannot very easily analyze the complex interactions between depot stock levels and expected base level performance.

Finally, the revised method makes it difficult to measure the effect of transportation policy. This occurs because daily fluctuations in expected performance are not considered in the optimization procedure.

The last model we will discuss is the one used by the Air Force to compute spare aircraft engine requirements for a wartime environment [2,3]. It is a stationary model. The daily demand rate at a location is increased to reflect the average daily flying activity there over the planning horizon. Using this average daily demand rate, the stock levels are computed separately for each location, thereby ignoring any depot-base interactions. Some of the pitfalls of using this model for determining wartime stock levels are illustrated in the example presented in Sec. II. As shown in that example, this approach makes it impossible to evaluate the dynamic behavior of the system. It also assumes that the output rates of the base and depot repair processes

immediately following the increase in flying activity instantaneously equal the wartime rates--an impossibility, since at best the output rate at time t from base maintenance must correspond to the input rate at time t minus the base repair time, B, that is, at time t-B. But the input rate at time t-B would most likely be considerably less than the input rate at time t. Thus the resupply rate during the critical early portion of a short war will most likely be overstated.

The models we have briefly discussed, and others like them, for computing item stock levels based on a nonstationary representation of the demand process would use primarily the same planning data that are required by current techniques, but would use these data more effec- tively. Consequently, the data needed are simply the dynamic base and depot repair times, NRTS rates, flying schedules, failure rates, and the transportation policy. Rather than aggregating flying and failure data over the planned horizon, we can obtain more precise estimates of the inventory requirements and supply effectiveness throughout the horizon by using the projected daily flying activity.

In addition to the application we have discussed, dynamic models can also be used effectively during the initial provisioning process to determine both when inventory should be added to the system and how much inventory is needed at each point in time to achieve a desired level of supply support. The initial provisioning techniques currently used by the Air Force are steady-state approaches. For example, the standard method for computing requirements is described in AFLCR 57-27 [10]. It is a simple deterministic model in which both the peak and average daily flying programs are used to forecast demand, and hence requirements, for

-45-

each item. This forecasted demand is implicitly assumed to occur uni-
formly at a constant rate during a year, even though demand is random
and the rate changes dramatically as additional aircraft are placed into
service. For basically the same reasons that we have already discussed,
this model inadequately addresses the dynamics of the demand process for
recoverable items of a new system over a period of time. Hence, it
gives an inaccurate portrayal of both expected supply support (such as
NMCS rates, fill rates, backorders, etc.) and inventory requirements.
This is illustrated in the following example.

Suppose a new aircraft system is introduced into the Air Force over
a two-year period. Furthermore, suppose the solid line in Fig. 4 is the
graph of the expected number of units in resupply in the system at each
point in time throughout the two-year horizon. The dashed lines
represent the number of units in resupply in the system assumed by the
AFLCR 57-27 model. Since the average number of units in resupply deter-
mines the stockage requirements, excess inventories will be on hand for
certain portions of the two-year period, while at other times the system
will experience severe shortages. This, of course, makes the planned
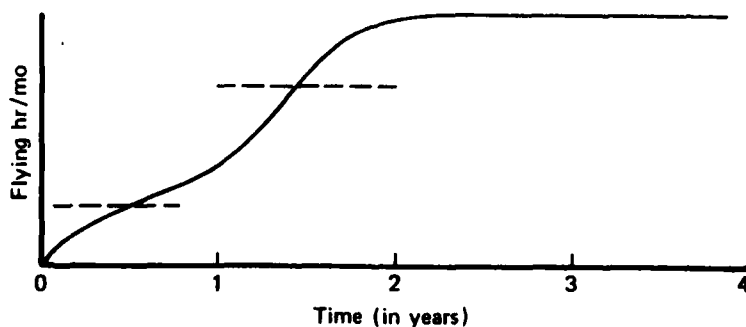flying program extremely difficult to achieve.



Fig. 4 — Expected number of units in resupply in the system

Rather than using a stationary model to describe demand during the provisioning process, dynamic models like the one developed in Sec. III can be used. The data required include the aircraft delivery and flying schedules, base and depot repair times, NRTS rates, failure rates, and order-and-ship-times. The only difference in data requirements is that the expected demand rate has to be expressed as a function of time. But this can be easily done using a computer. Once this nonstationary representation of the demand process is available, a model can be used to ascertain the optimal delivery schedule of spares for each item.

A model could be developed that determines the stocks that should be delivered so that the total cost of spares procurement is minimized subject to aggregate supply effectiveness constraints for each period (a month or quarter) of the initial provisioning planning  rizon. The cost could represent the discounted cost of purch ,ing s, .res over the planning horizon. Each constraint could, for example, establish a max-imum expected number of shortage days that would be allowed during a specific time period. Thus, the effect of different support goals for different periods could be explicitly examined using the model, and the effect on procurement cost could be measured directly.

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